Simplification of Boltzmann Equation on $S^3(1)$

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Abstract

Simple form of Boltzmann equation will be proposed after introducing a three-dimensional closed Lie group to simplify its collision term.

Keywords: molecular collision, Boltzmann equation, Lie Group

1 Introduction

Navier-Stokes equation is the first order approximation of Boltzmann equation[1]. Therefore, solving Boltzmann equation directly is useful to understand mysteries of fluid phenomena in detail[2]. The collision term in Boltzmann equation is probably the main difficulty, and the traditional treatments are limited in Perturbation, Variation, BGK methods and so forth[3], among which BGK model is the most successful to hydrodynamics; however, this method is still limited in treating fluid dynamics with constant temperatures and low Mach number[4].

On the other hand, Lie group and Lie algebra, original in analyzing Partial Differential Equations from the point of mathematics, is also used to deal with Boltzmann equation. However, the corresponding solutions only exist locally, and some of them are sensitive to the symmetrical structures as shown in[5, 6, 7, 8, 9]

To overcome such difficulties, a three-dimensional closed Lie group $S^3(1)$ imbedded in $\mathbb{R}^4$, on which the collision term can be dismissed, is introduced in this paper, and the global solution of Boltzmann equation is discussed.

2 Analysis of Collision Term and Results

Two molecules are m_1 and m_2 in mass, d_1 and d_2 in diameter, $\mathbf{v}_1$ and $\mathbf{v}_2$ in velocity before collision, and $\mathbf{w}_1$ and $\mathbf{w}_2$ after collision, respectively. The collision between molecules is imperfect elastic. In terms of momentum theorem and energy conservation law, we have

$$\begin{cases} m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{w}_1 + m_2\mathbf{w}_2 \\ \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 = \frac{1}{2}m_1\mathbf{w}_1^2 + \frac{1}{2}m_2\mathbf{w}_2^2 + \triangle E \end{cases} \tag{2.1}$$

where

$$\triangle E = \frac{1}{2}(1-\epsilon^2)\frac{m_1 m_2}{m_1+m_2}(v_1-v_2)^2 \tag{2.2}$$

and ϵ is the restitution coefficient. Define $\mid \mathbf{d} \mid$ the distance between centers of $m_1$and m_2. Let $\mathbf{n} = \frac{\mathbf{d}}{|\mathbf{d}|}$ and

$$\begin{cases} \mathbf{w}_1 - \mathbf{v}_1 = \lambda_1 \mathbf{n} \\ \mathbf{w}_2 - \mathbf{v}_2 = \lambda_2 \mathbf{n} \end{cases} \tag{2.3}$$

From Eq.2.1 and Eq.2.3, two cases of collision are deduced as

$$\begin{cases} \mathbf{w}_1 = \mathbf{v}_1 + (1+\epsilon)\frac{m_2}{m_1+m_2}[(v_2-v_1)\cdot\mathbf{n}]\mathbf{n} \\ \mathbf{w}_2 = \mathbf{v}_2 - (1+\epsilon)\frac{m_1}{m_1+m_2}[(v_2-v_1)\cdot\mathbf{n}]\mathbf{n} \end{cases} \tag{2.4}$$

and

$$\begin{cases} \mathbf{w}_1 = \mathbf{v}_1 + (1-\epsilon)\frac{m_2}{m_1+m_2}[(v_2-v_1)\cdot\mathbf{n}]\mathbf{n} \\ \mathbf{w}_2 = \mathbf{v}_2 - (1-\epsilon)\frac{m_1}{m_1+m_2}[(v_2-v_1)\cdot\mathbf{n}]\mathbf{n} \end{cases} \tag{2.5}$$

both of them may exist in experiment as shown in [1]. Write the collision term of Boltzmann equation as[1]

$$\frac{\partial f_1}{\partial t} \mid_{coll} = \iint (J^* f_1^{'} f_2^{'} - f_1 f_2)\frac{1}{4}d^2(\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{n}d\mathbf{v}_2 d\Omega \tag{2.6}$$

where $f(\mathbf{r},\mathbf{v},t)$ is an one-particle probability distribution function; $f,\ f^{'}$ denote the one-particle probability distribution function before and after collision. Ω is the scattering angle of the binary collision $\{\mathbf{W}_2, \mathbf{W}_1\} \to \{\mathbf{v}_2, \mathbf{v}_1\}$. Here J^* is the Jacobean matrix defined as

$$J^* = \frac{\partial(\mathbf{W}_2, \mathbf{W}_1)}{\partial(\mathbf{v}_2, \mathbf{v}_1)} \tag{2.7}$$

In terms of Eq.2.4 and Eq.2.5

$$J^* = \epsilon \tag{2.8}$$

Write Eq.2.6 in general form as matter of convenience

$$\frac{\partial f}{\partial t} \mid_{coll} = \iint (\epsilon f^{'} f_1^{'} - f f_1)\frac{1}{4}d^2(\mathbf{v} - \mathbf{v}_1)\cdot\mathbf{n}d\mathbf{v}_1 d\Omega \tag{2.9}$$

In order to treat the above collision term, we shall first introduce a four-dimensional Euclidean space. Let $\lambda \in \mathbb{R} \setminus \{0\}$, $(v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, such that

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 = \lambda^2 \tag{2.10}$$

then the following differentiable manifold

$$M = S^3(1) = \{(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} \vartheta_i^2 = 1\}$$

is a Lie group[10], where $\vartheta = \frac{\mathbf{v}}{\lambda}$.

According to [11], Boltzmann equation can be written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = \frac{\partial f}{\partial t} |_{coll} \tag{2.11}$$

Recall the General Stokes Formula[12]

$$\int_{\partial M} F = \int_M dF \tag{2.12}$$

where the boundary ∂M of M is smooth and simple, $F \in C^\infty(M)$, we can obtain

$$\begin{aligned} \frac{\partial f}{\partial t} |_{coll} &= \int_s [\int_{\partial M} (\epsilon f' f_1' - f f_1) \frac{1}{4} d^2 (\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n} d\mathbf{v}_1] d\Omega \\ &= \int_s [\int_M \frac{\partial [(\epsilon f' f_1' - f f_1) \frac{1}{4} d^2 (\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n}]}{\partial v_4} dv_4 d\mathbf{v}_1] d\Omega \end{aligned} \tag{2.13}$$

Here we can see the collision term on the Lie group M . Suppose $(\epsilon f' f_1' - f f_1) \frac{1}{4} d^2 (\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n}$ to be smooth. Since it is independent to v_4, then we have

$$\frac{\partial f}{\partial t} |_{coll} = 0 \tag{2.14}$$

Consequently, the Boltzmann equation can be written as Vlasov-Poisson equation[13]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = 0 \tag{2.15}$$

Suppose e the identity in M, let $p = (0,0,0,1)$, $U = S^3(1) \setminus \{p\}$. Define $\varphi : U \longrightarrow \mathbb{R}^3$, thus (U, φ) is a chart of M containing identity e. where

$$\varphi(v_1, v_2, v_3, v_4) = (v_1^*, v_2^*, v_3^*) = (\frac{v_1}{1 - v_4}, \frac{v_2}{1 - v_4}, \frac{v_3}{1 - v_4}) \tag{2.16}$$

Here let $\mathbf{v} = \vartheta$ as a matter of convenience.

Eq.15 can be easily written in the form of

$$\frac{df}{dt} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = 0 \tag{2.17}$$

where $\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$, and from Eq.2.16

$$\frac{\partial}{\partial v_i} = \frac{\partial}{\partial v_j^*} \frac{\partial v_j^*}{\partial v_i} \tag{2.18}$$

Let $J = | \frac{\partial v_j^*}{\partial v_i} |$, and write Eq.2.15 as

$$\frac{df}{dt} + J \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}^*} f = 0 \tag{2.19}$$

Let $G(\tau)$ be the one-parameter subgroup on Lie group M, and define

$$X = \frac{J F_i}{m} \frac{\partial}{\partial v_i^*}$$

which is treated as Lie algebra on M; usually, we suppose constant acceleration of molecules[11] and the external force $\mathbf{F}$ to be independent with time t such

as gravitation. Recall the definition of tangent vectors on manifold and the character of one-parameter subgroup[10, 14]

$$dG(\frac{d}{d\tau}(0))f = \frac{d}{d\tau}(f \circ G)(0) = Xf(e) \tag{2.20}$$

Here we suppose $f \in C^{\infty}(e)$, $f \circ G \in C^{\infty}(0)$. Then Eq.2.18 can be written as

$$\frac{df}{dt}(e) + \frac{d}{d\tau}(f \circ G)(0) = 0 \tag{2.21}$$

Since M is a global Lie group, we can write the above equation in the form of

$$\frac{df}{dt} + \frac{d}{d\tau}(f \circ G) = 0 \tag{2.22}$$

Moreover, let $\tau = \theta(t)$, and $d\tau = \theta^{'}(t)dt$, then integrate Eq.2.22

$$f(\mathbf{v}, t) + \theta^{'}(t)f(G(\theta(t))) = C(\mathbf{v}) \tag{2.23}$$

where $C(\mathbf{v})$ is independent with t, and determined by boundary and initial conditions; $G(t)$ is a known function.

3 Conclusion

We firstly derived the collision term in a general process with restitution coefficient. However, this coefficient makes no difference to ours analysis, for the collision term in Boltzmann equation is bound to disappear as long as it is on a closed differentiable manifold. At the same time, we can always introduce a higher dimensional space for any $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, such that

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 = \lambda^2$$

where the parameter λ controls the solution of Boltzmann equation, and manifold $S^3(1)$ is a global Lie group. So the differentiable manifold is well-defined and the corresponding results are global.

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